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Diffraction by a Wide Slit

S. N. KARP and A. RUSSEK

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Abstract

The problem of diffraction of scalar waves by an infinite conducting plane with a slit is investigated. Approximate expressions for the near and far fields, taking into account the interaction between the edges, are derived in terms of the well-known solutions for the field produced when an isolated conducting half-plane is excited by a) a plane wave, and b) a line source. Results of numerical calculation are given for the case of a plane wave normally incident on the slit. Twelve values of slit width ranging from .96 to 2.5 wavelengths are considered. For each slit width the far diffracted field is computed at ten-degree intervals in our approximation, and is compared with the exact solution obtained by Mathieu functions, and with the results obtained when interaction is neglected. A comparison of transmission coefficients is also given. It is found that the new approximate solution agrees well with the exact solution, and provides a significant correction to the non-interaction solution. The accuracy increases with the slit width, so that the result is useful in the range where interaction cannot well be neglected but where the exact solution converges so slowly that computation is impracticable. A brief discussion of the case of line-source excitation is included.

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I. Introduction

Since there already exist numerous investigations of the diffraction of scalar waves by a slit, the addition of another item to the literature requires some explanation. The purpose of the present note is to give an elementary and intuitive account of the diffraction by an infinite conducting plane with a slit. We consider specifically the case of electromagnetic waves whose electric field possesses only a single component, namely a component in the direction parallel to the edges of the slit. The results are also applicable to the acoustic problem, for soft screens. It is found that the results for large slit widths can be expressed approximately in terms of simple well-known functions and that these approximations are more accurate than those conveniently obtainable by previous procedures.

The procedures that have hitherto been proposed and applied to this problem fall into three categories. First, there is the exact theory based upon separation of variables in elliptic coordinates[1,2,3]. This procedure is useful only when the ratio of slit width to wavelength is small (that is, say, smaller than 3). Second, there are approximation procedures that are valid when the ratio of slit width to wavelength is small. Such procedures, based on the solution of related electrostatic problems, were indicated by Rayleigh[4] and have been developed further by, e.g., Sommerfeld[5], Groschwitz and Hönl[6], and Bouwkamp[7]. Finally, there are approximate treatments valuable particularly for large ratios of slit widths to wavelength; among these are calculations based on Kirchoff's procedures and their various modifications, and the work of Schwarzschild[8]. A calculation related to that of Schwarzschild has recently been published by Fox[9] in connection with diffraction by a pulse. It should be noted that variational techniques may be applied in conjunction with any of the approximate methods just mentioned.

The procedure adopted here is closely related to the work of Schwarzschild, who indicated a series of successive approximations, based on the Sommerfeld solution of the problem of diffraction by a half-plane. Schwarzschild considers the slit configuration as being composed of two diffracting half-planes. Each half-plane responds to a variety of excitations, and the response in each case can be calculated by use of the Sommerfeld Green's function for the half-plane. The excitations in question are calculated successively, as are the responses. The first excitation is the incident field, and the corresponding first approximation to the total field is the sum of the fields diffracted independently by each half-plane, according to Sommerfeld, under this excitation. The second excitation for the left half-plane, say, is the field scattered by the right half-plane, and so on. This leads to an infinite series of terms, which Schwarzschild shows, is convergent for every slit width. However, this method is most valuable for large separation of the half-planes. Schwarzschild calculated explicitly only the first term described above, in which the excitation for each half-plane is the incident wave only. This term will henceforth be called the 'non-interaction approximation'.

The investigation pursued in the present paper is based on a similar conception of the action of a slit, but the procedure used here differs somewhat from that of Schwarzschild. The field at a point is considered as being composed of the incident field plus a response field from each of the two half-planes, or alternatively, as a geometric optics field plus two diffracted fields (one from each half-plane). It is assumed that each of these diffracted fields is qualitatively of the same nature as the known field diffracted by an isolated half-plane. In particular, it is supposed that the partial field, i.e., the field ascribed to one of the half-planes, say the right half-plane, can be well represented in the region far enough to the left by a function of the form

$(e^{-ikr}/\sqrt{r})f(\theta)$ (i.e., the form of the function representing the field diffracted by an isolated half-plane); here r and θ are polar coordinates with pole at the edge of the right half-plane. However, the angular function $f(\theta)$, known for diffraction by an isolated half-plane, must be determined for the case of diffraction by a slit. Now on the left half-plane, θ has the constant value π. Thus on the left half-plane, say, the field due to the right half-plane is everywhere of the form const. $\chi(e^{-ikr}/\sqrt{r})$; and this represents, approximately, the field of a line source situated at the edge of the right half-plane. A similar remark holds for the net excitation of the right half-plane by the net field ascribed to the left half-plane. Hence each half-plane must be thought of as being excited by the incident field plus a line source located at the edge of the opposite half-plane; the strength of this line source depends on the value of $f(\pi)$ for the opposite half-plane. The total field can therefore be expressed as a linear combination, with constant coefficients, of fields of a well-known and simple form, i.e., the fields which occur in diffraction of plane-wave and line-source fields by half-planes. The coefficients are calculated by comparing, for each half-plane, the value of $f(\pi)$ as an excitation and as a response; this yields two linear algebraic equations for the two coefficients involved.

As a result of this approximate calculation, the diffracted field is expressed as the sum of a 'non-interaction' term plus an 'interaction' term, the non-interaction term being identical with Schwarzschild's leading term. Our interaction term, conveniently expressed in terms of the response of a half-plane to a line source of a prescribed amplitude placed at the opposite edge, has a form presumably different from the function that would arise when calculating Schwarzschild's second term. This difference would probably be a result of approximating the same function by different methods, and involves terms of higher order in the parameter, i.e., the slit width divided by wavelength. It seems possible that the interaction term derived here is more reliable since it partially takes into account the total interaction. It is evident from the method used that the result derived here is at best an asymptotic approximation and that it can be expected to be good when the ratio of slit width to wavelength is large, and further, that for any such value of this parameter, it should be an improvement over Schwarzschild's first term. This was found to be so in the cases for which calculations were made.

In Section II the relevant known formulas dealing with diffraction by a half-plane are reviewed and developed in a suitable form. Section III is devoted to the calculation described above, for the case of a plane wave incident from an arbitrary direction, and in Section IV we deal with the special case of normal incidence. Numerical computations were carried out for twelve values of the ratio of slit width to wavelength in the range in which this parameter varies between ~ 1 and ~ 2.5. This range substantially represents the overlap between the upper limit for which the exact calculations can be conveniently handled and the lower limit to which the asymptotic approximation derived here can be pushed. In each case, the far transmitted field is calculated in ten-degree intervals for a) the exact calculation, b) the non-interaction approximation, and c) the approximation including the interaction term derived here. Although the calculations for the exact field have been done previously[1,2,3], it was found desirable to carry out these calculations independently using the Bureau of Standards Tables relating to Mathieu Functions[10].

The numerical results are presented in Tables I and II, the field strengths being listed in the former, the transmission coefficients in the latter. It is seen from Table I that the interaction correction derived in this paper significantly improves the non-interaction approximation, and it seems plausible that this correction term may be used with confidence for all slits more than one wavelength in width.

In Section V, the field transmitted by a slit when excited by a line source is considered, while Section VI is devoted to concluding remarks.

II. Diffraction by a single half-plane

In this section we review the well-known solution of the problem of diffraction of a line source or plane wave by a conducting half-plane, and take special notice of certain simple properties which will prove useful in the later sections. We consider E polarization, with an electric field whose z-component we call E_z. The latter must vanish on the screen.

The field of a line source in the presence of a conducting half-plane whose edge is parallel to the source is well-known. Suppose the line source is parallel to the z-axis and that its field, in the absence of the half-plane, is

$$E_z = \frac{\pi}{2i} H_o^{(2)}(kR). \tag{1}$$

Then the field in the presence of the half-plane is given by the integral expression[11]

$$E_z = \frac{1}{2} \left\{ \int_{-\infty}^{\xi_o} e^{ikR \cosh \xi} \, d\xi - \int_{-\infty}^{\xi_1} e^{-ikR_1 \cosh \xi} \, d\xi \right\} , \tag{2}$$

where

$$\begin{aligned} \sinh \xi_o &= \frac{2\sqrt{rr_o}}{R} \cos \tfrac{1}{2} (\theta - \theta_o), \\ \sinh \xi_1 &= \frac{2\sqrt{rr_o}}{R_1} \cos \tfrac{1}{2} (\theta + \theta_o), \\ R &= \sqrt{(x-x_o)^2 + (y-y_o)^2}, \\ R_1 &= \sqrt{(x-x_o)^2 + (y+y_o)^2} . \end{aligned} \tag{3}$$

The coordinates are defined in Fig. 1.

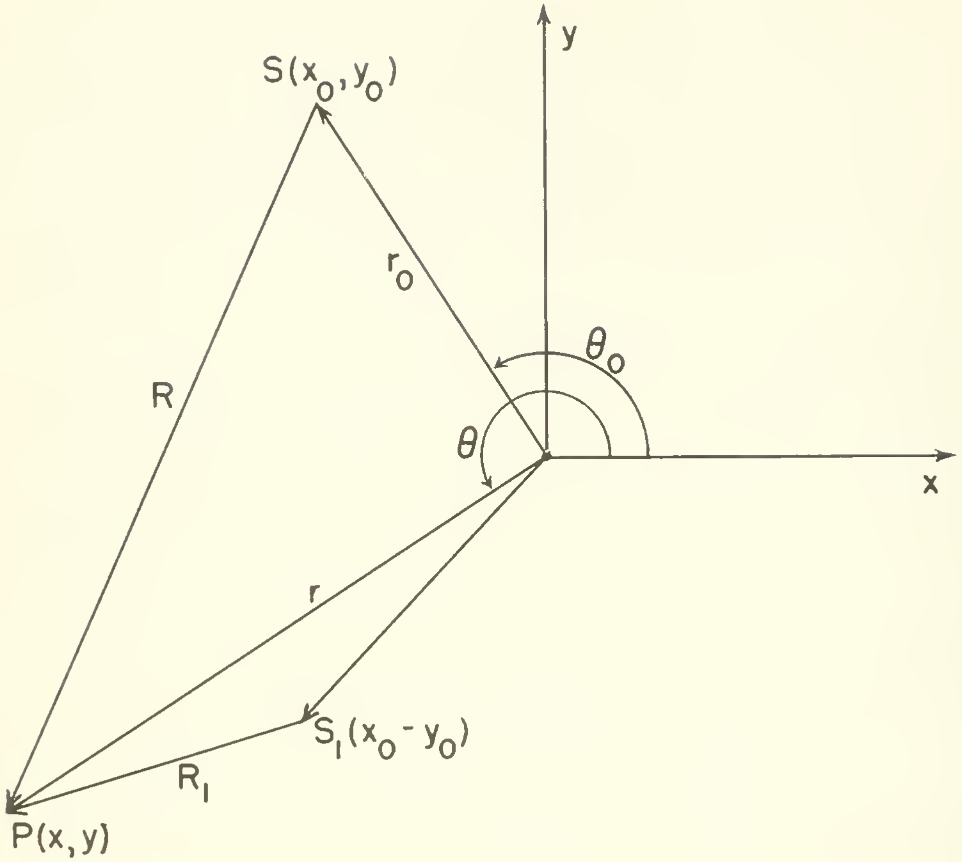

Figure 1

The half-plane covers that portion of the plane $y = 0$ for which $x > 0$, and its edge is coincident with the z-axis. The source S is located at (x_o,y_o) or, in polar coordinates, (r_o,θ_o), and the field point $P(x,y)$ is denoted, in polar coordinates, by (r,θ). R is the distance from the source S to the field point P; R_1 is the distance from S_1 to P, S_1 being the image of S mirrored in the plane of the half-plane.

The far field (that is, the field where $kr \gg 1$) is approximately of the form

$$E_z \simeq E_z^{(g)} + \sqrt{\frac{\pi}{2i}}\ \frac{e^{-ikr}}{\sqrt{kr}}\ f(r_o,\theta_o,\theta), \tag{4}$$

except in certain regions shortly to be described. An alternative description is

$$E_z = E_z^{inc} + \sqrt{\frac{\pi}{2i}}\,\frac{e^{-ikr}}{\sqrt{kr}}\,F(r_o,\theta_o,\theta). \tag{4a}$$

Here $F(r_o,\theta_o,\theta)$ is expressible in Fresnel integrals. The expression (4a) is valid for all θ and sufficiently large r. Eqs. (4) and (4a) can be derived from Eq. (2). In Eq. (4), $E_z^{(g)}$ is the field calculated on the basis of geometric optics and is specified in terms of three distinct regions, as illustrated in Figure 2 for a particular case. Region I is the geometric shadow of the half-plane, and in this region $E_z^{(g)} = 0$; II is illuminated by direct rays only, and here $E_z^{(g)} = \frac{\pi}{2i} H_o^{(2)}(kR)$; and Region III is illuminated by both direct and reflected rays, so that

$$E_z^{(g)} = \frac{\pi}{2i}\left\{ H_o^{(2)}(kR) - H_o^{(2)}(kR_1)\right\} .$$

The lines separating the regions are known as 'shadow lines', and on them $E_z^{(g)}$ is discontinuous. The angular neighborhoods of these lines are the exceptional regions we mentioned in connection with Equation (4).

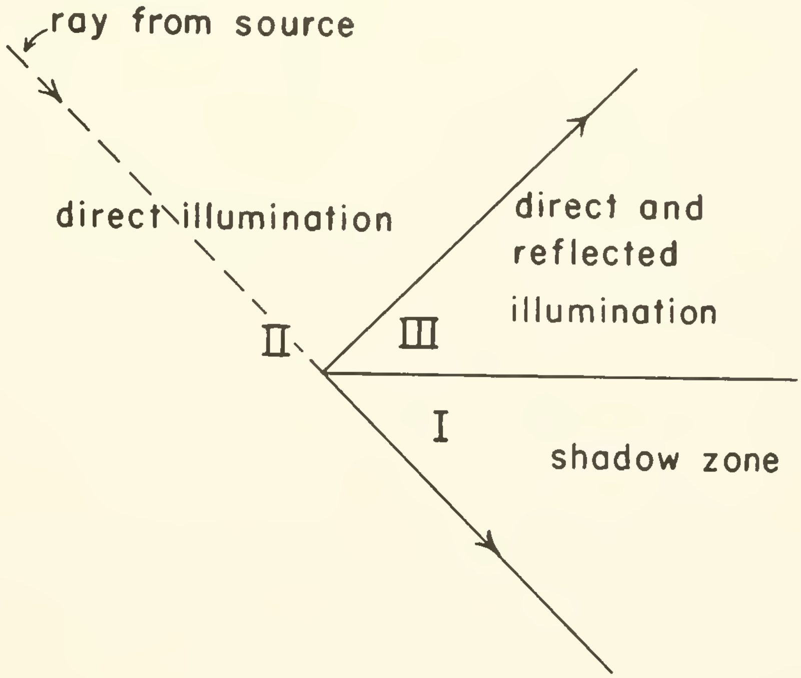

Figure 2

We shall define $E_z^{(d)}$ to be the diffracted field, everywhere given by

$$E_z = E_z^{(g)} + E_z^{(d)} . \tag{4b}$$

The second term on the right side of Eq. (4) is then a far-field approximation to $E_z^{(d)}$, and since for kr >> 1,

$$\sqrt{\frac{\pi}{2i}}\ \frac{e^{-ikr}}{\sqrt{kr}} \simeq \frac{\pi}{2i}\ H_o^{(2)}(kr), \tag{5}$$

the far diffracted field is approximately that which would be produced by an 'inhomogeneous' line source placed at the edge of the half-plane.

For the present, it is necessary to have an expression for the line-source field only for the case where the half-plane is excited by a line source located on the extension of the half-plane (i.e., $\Theta_o = \pi$). In this case both the shadow and reflected zones (Regions I and III) disappear, so that $E_z^{(g)}$ is just the incident field and f is given by

$$f(r_o,\pi,\Theta) = e^{-ikr_o\cos\Theta}\left\{-1 + \sqrt{2i}\int_0^{2\sqrt{kr_o/\pi}\,\sin(\Theta/2)} \exp(-i\pi v^2/2)dv\right\}, \tag{6}$$

as can be seen by evaluating (2) for large kr. When

$$2\sqrt{\frac{kr_o}{\pi}}\ \sin\frac{\Theta}{2} >> 1, \tag{7}$$

then (6) can be approximated by

$$f(r_o,\pi,\Theta) \simeq - \frac{e^{-ikr_o}}{\sqrt{2\pi ikr_o}\ \sin\frac{\Theta}{2}} . \tag{8}$$

From (7) is is clear that the approximation (8) is valid only if kr_o is large compared to unity, and furthermore if the angle Θ is not close to zero or 2π (i.e., the field point must not be near the half-plane).

When the incident field is not the line source (1) but instead a plane wave of unit amplitude approaching from the direction θ_o, i.e., when

$$U_{\text{incident}} = e^{-ikr\cos(\theta-\theta_o)}, \tag{9}$$

then the far field ($kr \gg 1$) in angular regions bounded away from the shadow line is given approximately by

$$E_z \simeq E_z^{(g)} + \sqrt{\frac{\pi}{2i}}\,\frac{e^{ikr}}{\sqrt{kr}}\; g(\theta_o,\theta), \tag{10}$$

where the geometrical optics term $E_z^{(g)}$ is now zero in Region I; $\exp\left[-ikr\cos(\theta-\theta_o)\right]$ in Region II; and $\exp\left[-ikr\cos(\theta-\theta_o)\right] - \exp\left[-ikr\cos(\theta+\theta_o)\right]$ in Region III. Away from the shadow lines, the angular function $g(\theta_o,\theta)$ is given by

$$g(\theta_o,\theta) = -\frac{1}{2\pi}\left\{\frac{1}{\cos\frac{1}{2}(\theta-\theta_o)} - \frac{1}{\cos\frac{1}{2}(\theta+\theta_o)}\right\}. \tag{11}$$

Thus, like the line-source field, the field produced when a half-plane is excited by a plane wave can be expressed as the sum of a geometrical optics term plus a diffraction term whose far-field approximation has the form of an inhomogeneous line source located at the edge of the half-plane.

III. Diffraction by a slit

Since an infinite conducting plane with a slit is a pair of adjacent half-planes, an approximate expression for the field transmitted by such a slit as a result of excitation by a plane wave can be obtained in terms of the well-known results of the previous section. As before, the field may be expressed as the sum of a geometric optics term plus a diffraction term:

$$E_z = E_z^{(g)} + E_z^{(d)}. \tag{12}$$

In the three regions required to specify $E_z^{(g)}$, shown in Figure 3, the geometrical optics fields are, respectively: zero in Regions Ia and Ib, the incident wave in Region II, and the incident wave plus reflected wave in Region IIIa and IIIb. The two-half-planes are denoted by the subscripts a and b,

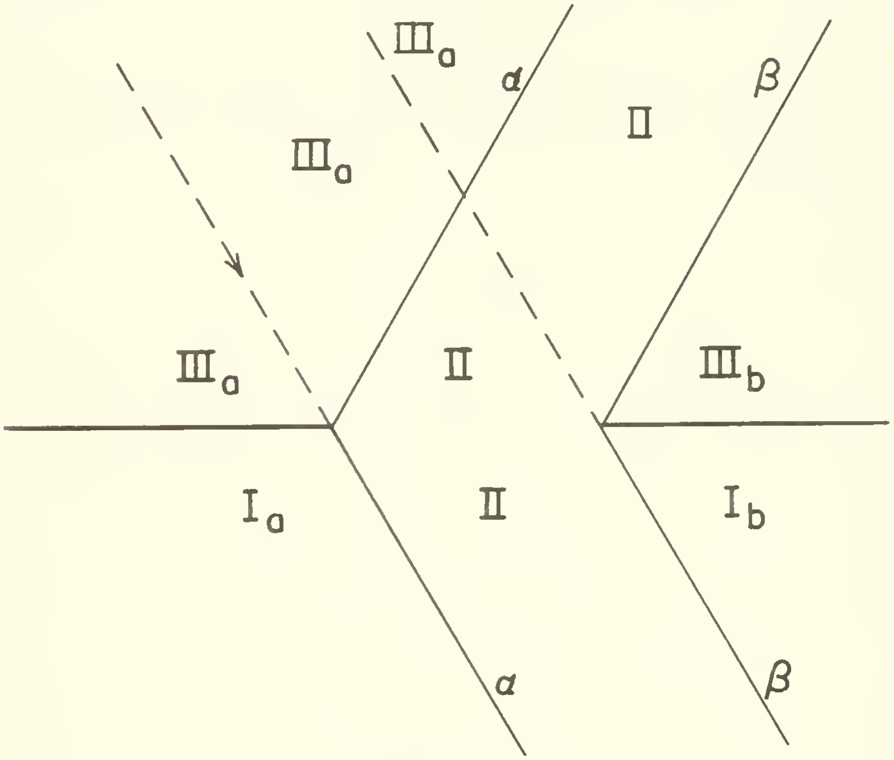

Figure 3

and to simplify the mathematics two coordinate systems are employed, for the time being, with origins at the edges of these half-planes (see Figure 4). The plane wave is incident from directions θ_{ao} and θ_{bo} in the two coordinate systems respectively, while the coordinates of the field point are (r_a,θ_a) and (r_b,θ_b) respectively. Taking the slit width to be 2d (d is what is commonly called the ('half-width'), each edge is located at $(2d, \pi)$ with respect to the coordinate system of the opposite edge.

The mathematical formulation of the problem is as follows. Given an incident field $U^{inc}(x,y) = E_z^{inc}$, we have to determine a field $U(x,y) = E_z$, in accordance with the conditions that

(i) $\nabla^2 U + k^2 U = 0$ (except on the half-planes (a) and (b));

(ii) $U(x,y) - [U^{inc}(x,y) - U^{inc}(x,-y)] \quad r^{-1/2}\exp(-ikr)f_1(\theta)$, $(y > 0)$;

(iii) $U(x,y) \sim r^{-1/2} \exp(-ikr) f_2(\theta)$, $(y < 0)$;

(iv) $U = 0$ at the surfaces (a) and (b).

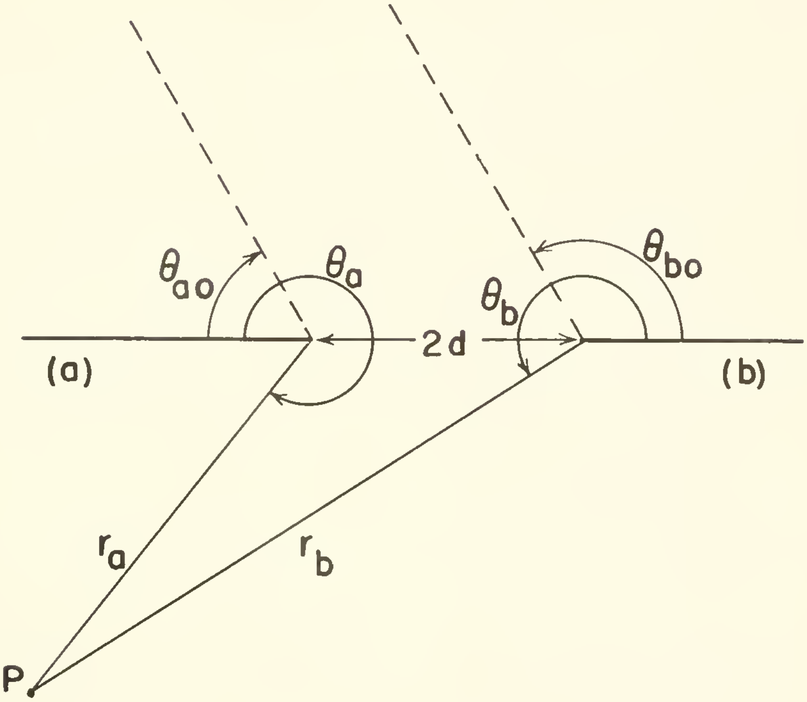

Figure 4

Here r and θ are polar coordinates with pole at the center of the slit. For $y > 0$ we have $0 \leq \theta \leq \pi$, while for $y < 0$, we have $\pi \leq \theta \leq 2\pi$. Conditions (ii) and (iii) are to hold for large kr uniformly in θ in the intervals $(0 \leq \theta \leq \pi)$ and $(\pi \leq \theta \leq 2\pi)$ respectively.

Let us now write the solution in the form

$$U = U^{inc} + U_a^r + U_b^r .$$

The superscript r denotes 'response'. We are here trying to express U in the form of an incident field plus two response fields, each originating in the half-plane indicated by its subscript. We assume therefore that U_a^r is regular in the exterior of surface (a), and U_b^r is regular in the exterior of surface (b).

We now make some further assumptions as to the nature of U_a^r and U_b^r, and then try to determine these functions approximately. The success of this procedure will then depend on the degree to which the regularity and boundary conditions are fulfilled by the resulting approximation to the function U.

We assume that the response fields U_a^r, U_b^r have respectively the same behavior as the known response which half-planes (a) and (b) would make to the incident field if each were isolated. To express this assumption mathematically we first define functions E_z^{da} and E_z^{db} by the following equations (cf. Fig. 3):

$$
\begin{aligned}
U^r_{a,b} &= -U^{inc}(x,-y) + E_z^{da,db} && \text{in IIIa, IIIb;} \\
U^r_{a,b} &= -U^{inc}(x,y) + E_z^{da,db} && \text{in Ia, Ib;} \qquad (12a) \\
U^r_{a,b} &= E_z^{da,db} && \text{otherwise.}
\end{aligned}
$$

The above equations are to be read twice, once as applying to U_a^r, E_z^{da} and regions IIIa and Ia, and again as applying to U_b^r, E_z^{db} and regions IIIb and Ib. Then the assumption of a response behavior analogous to the isolated half-plane responses is seen to be equivalent to the postulation of the asymptotic formulas

$$
E_z^{da,db} \sim (r_{a,b})^{-1/2} \exp(-ikr_{a,b}) F_{a,b}(\theta_a,\theta_b). \qquad (12b)
$$

These formulas are assumed to be valid for large kr_a or kr_b respectively, in angular regions bounded away from the shadow lines. Thus for E_z^{da}, Equation (12b) is assumed to be valid for θ_a bounded away from the lines labeled α in Figure 3. For E_z^{db} the lines labeled β are excluded. The behavior implied in Equations (12a) and (12b) actually occurs in the case of a single half-plane. Of course we are not assuming that the functions $F_a(\theta_a)$ and $F_b(\theta_b)$ are the same as those for an isolated half-plane.

The assumptions we have made thus far can be summarized by saying that we have assumed that the function U defined in our mathematical formulation can be represented in the form $U = U^{inc} + U_a^r + U_b^r$, where the functions U_a^r and U_b^r are restricted by the requirements expressed in (12a) and (12b) and by the requirements that $U_{a,b}^r$ be regular everywhere except on the surfaces (a) and (b) respectively.

It should be remarked here that the crux of our argument, and the approximation involved in it, has not yet appeared. In fact, the physically plausible assumptions made so far can actually be proven rigorously; they are mathematical consequences of the conditions of the problem. Although it is sufficient to make these physical assumptions outright, since when we obtain our approximate solution we shall verify them in any case, it seems useful to indicate a proof here. This would consist of expressing the response to U^{inc}, by means of Green's theorem, in the form of two integrals whose surfaces of integration are (a) and (b) respectively. Each of these integrals is a regular wave function outside its surface of integration. The integrals represent distributions of line sources. Since the density of distribution, or current density, can be expressed in terms of the normal derivative of U, its asymptotic behavior can be shown [using the conditions (i) - (iv)] to approach a value equal to twice the normal derivative of the incident field. From this one can show that the surface integrals we have described behave for large kr_a, kr_b in the way we have postulated for our functions U_a^r and U_b^r. Thus the Green's function integrals actually provide a representation of the exact solution in the form we have assumed.

We now proceed with the essentially approximate part of our argument. From equation (12a) we know that near half-plane (a), the function U_b^r is given by E_z^{db}. The latter function, in turn, is now seen to be equivalent (near half-plane (a)) to some constant multiple of a line source $H_o^{(2)}(kr_b)$, provided the slit is large compared to wavelength. This follows from equation (12b) if we note that the function $r_b^{-1/2} \exp(-ikr_b)$ is a constant multiple of the function

$H_o^{(2)}(kr_b)$, for large kr_b, and also that along half-plane (a) we have $\theta_o = \pi$ and $F_b(\theta_b) = F_b(\pi) =$ constant. We now observe that the meaning of the above result is that half-plane (a) is being excited, in effect, by the incident field and, in addition, by a line source of undetermined strength located at the edge of half-plane (b). We argue that the response of (a) must be equal to the known response of an isolated half-plane to the incident field plus some as yet undetermined multiple of the known response which such a half-plane makes to a unit line source. Similarly we find that half-plane (b) is being excited by a line source located at the edge of (a), in addition to the given incident excitation. We are therefore led to the conclusion that U_b^r can similarly be expressed as a linear combination of the response to the incident field and the response to a line source. Let U_{ai}^r be the known response of half-plane (a) to the given incident field when half-plane (b) is absent, and let U_{ab}^r be the known response of the half-plane (a) to a line source $\frac{\pi}{2i} H_o^{(2)}(kr_b)$ again when (b) is absent. This line source is located at $r_a = 2d$, $\theta_a = \pi$ in the r_a, θ_a coordinate system. We have seen that U_b^r has the form const.$\times \frac{\pi}{2i} H_o^{(2)}(kr_b)$ at points on half-plane (a). Let C_a be the unknown constant involved. Also introduce the known functions U_{bi}^r, U_{ba}^r, and the unknown constant C_b in a similar manner: U_{bi}^r is the response of isolated half-plane (b) to the incident field, U_{ba}^r is the response to the excitation $\frac{\pi}{2i} H_o^{(2)}(kr_a)$, and C_b is the constant involved in the asymptotic representation of U_a^r at points on half-plane (b). Then we may write

$$
\begin{aligned}
U_a^r &\cong U_{ai}^r + C_a U_{ab}^r , \\
U_b^r &\cong U_{bi}^r + C_b U_{ba}^r , \qquad (13) \\
U &\sim \left\{ U_{inc} + U_{ai}^r + U_{bi}^r \right\} + C_a U_{ab}^r + C_b U_{ba}^r .
\end{aligned}
$$

This representation holds at any point in space.* To complete the specification of U, U_a^r and U_b^r we must determine the unknown constants C_a and C_b. Before turning to this problem we consider the quantities E_z^{da}, E_z^{db} given in Eqs. (12b). These equations may now be rewritten in greater detail in view of Eqs. (13), if we use the asymptotic forms of the diffracted fields of the half-plane for plane-wave and line-source excitation. Using the notations of Section II, Eqs. (10) and (11), and Eqs. (4), (6), (7) and (8), we find

$$E_z^{da} \simeq \sqrt{\frac{\pi}{2i}}\, \frac{e^{-ikr_a}}{\sqrt{kr_a}} \left[g(\theta_{ao},\theta_a) + C_a f(2d,\pi,\theta_a) \right] , \tag{13a}$$

$$E_z^{db} \simeq \sqrt{\frac{\pi}{2i}}\, \frac{e^{-ikr_b}}{\sqrt{kr_b}} \left[g(\theta_{bo},\theta_b) + C_b f(2d,\pi,\theta_b) \right] . \tag{13b}$$

These formulas hold if we let r_a or r_b tend to infinity while θ_a (or θ_b) are bounded away from the lines α (or the lines β) shown in Figure 3.

Let us also note at this point a significant regrouping of Equations (13), (13a), and (13b). We can write

$$u \equiv E_z = E_z(\text{non}) + E_z(\text{int}) \tag{13c}$$

with

$$E_z(\text{non}) = U_{\text{inc}} + U_{ai}^r + U_{bi}^r \tag{13d}$$

and

$$E_z(\text{int}) = C_a U_{ab}^r + C_b U_{ba}^r . \tag{13e}$$

The non-interaction term $E_z^{(d)}(\text{non})$ is exactly the first term of the Schwarzschild series mentioned in Section I, and represents the excitation of each half-plane by the original plane wave only. It may be a good approximation for extremely wide slits, in which case the half-planes have little effect on each other. For narrower slits, where the interaction of the half-planes cannot be neglected, the correction terms $E_z^{(d)}(\text{int})$ is added.

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* It is to be noted that it is an approximate representation in terms of known functions.

If we take our field point to infinity in the lower half-space in a fixed direction not parallel to the shadow lines α and β, then we enter either Region I_a or I_b. In Region I_a we have $U_a^r = -U^{inc} + E_z^{da}$ and $U_b^r = E_z^{db}$, while in Region I_b we have $U_b^r = -U^{inc} + E_z^{db}$ and $U_a^r = E_z^{da}$ [cf. Eq. (12a)]. Thus in either case we find that $U \equiv E_z$ is given by the sum of E_z^{da} and E_z^{db}, which are a pair of functions of the form indicated in (12b); these functions are now to be further specified. Furthermore in Regions I_a and I_b the geometrical optics field as we specified it in connection with Fig. 3 is zero, and hence the total field is also the diffracted field as defined in Eq. (12). Consequently, if we pass to infinity in the lower half-space, in a direction not parallel to a shadow line, the following formulas hold in virtue of Equations (13a) - (13e):

$$E_z \equiv E_z^{(d)} = E_z^{(d)}(\text{non}) + E_z^{(d)}(\text{int}), \tag{14}$$

where

$$E_z^{(d)}(\text{non}) \cong \sqrt{\frac{\pi}{2i}}\left\{\frac{e^{-ikr_a}}{\sqrt{kr_a}}\, g(\theta_{ao},\theta_a) + \frac{e^{-ikr_b}}{\sqrt{kr_b}}\, g(\theta_{bo},\theta_b)\right\} \tag{15a}$$

and

$$E_z^{(d)}(\text{int}) \cong \sqrt{\frac{\pi}{2i}}\left\{\frac{e^{-ikr_a}}{\sqrt{kr_a}}\, C_a f(2d,\pi,\theta_a) + \frac{e^{-ikr_b}}{\sqrt{kr_b}}\, C_b f(2d,\pi,\theta_b)\right\}. \tag{15b}$$

We now turn to the determination of the constants C_a and C_b. We have been regarding half-plane (a) as subject to the excitation of the incident field, and, in addition, to excitation by a fictitious line source located at the edge of (b). This line source gives the effect of the function U_b^r. Equation (13b) shows that near half-plane (a) we have $U_b^r \equiv E_z^{db}$; an expression for the latter is given in (13b). Since $(\pi/2ikr_b)^{1/2}\exp(-ikr) \sim (\pi/2i)H_o^{(2)}(kr_b)$, we see that

$E_z^{db} \sim \text{constant} \times (\pi/2i)H_o^{(2)}(kr_b).$

The constant of proportionality is (from (13b))

(16) $$g(\theta_{bo},\pi) + C_b f(2d,\pi,\pi).$$

Hence the partial far diffracted field produced by the half-plane (a) in specific response to the excitation by the fictitious line source at the edge of (b) is equal to the constant (16) times the well-known response [cf. (4)] to a homogeneous line source, and is given by

(17) $$\left[g(\theta_{bo},\pi) + C_b f(2d,\pi,\pi)\right]\sqrt{\frac{\pi}{2i}}\ \frac{e^{-ikr_a}}{\sqrt{kr_a}}\ f(2d,\pi,\theta_a).$$

Now from Eq. (13) we find a different expression for the partial field (17), namely

(18) $$C_a\sqrt{\frac{\pi}{2i}}\ \frac{e^{-ikr_a}}{\sqrt{kr_a}}\ f(2d,\pi,\theta_a).$$

Equating expressions (17) and (18), we have

(19) $$C_a = g(\theta_{bo},\pi) + C_b f(2d,\pi,\pi).$$

Proceeding as above with respect to the diffracted field produced by half-plane (b) due to excitation by the fictitious line source at the edge of (a), a similar relation is obtained:

(19a) $$C_b = g(\theta_{ao},\pi) + C_a f(2d,\pi,\pi).$$

Equations (19) and (19a) are two linear inhomogeneous equations in the desired coefficients and yield

(20) $$C_a = \frac{g(\theta_{bo},\pi) + g(\theta_{ao},\pi)f(2d,\pi,\pi)}{1 - [f(2d,\pi,\pi)]^2}$$

$$C_b = \frac{g(\theta_{ao},\pi) + g(\theta_{bo},\pi)f(2d,\pi,\pi)}{1 - [f(2d,\pi,\pi)]^2}\ .$$

The final formula for the field at any point is given, in terms of line-source and plane-wave fields, by equations (13) and (20).

In (20) we may substitute the approximation (8) for $f(2d,\pi,\pi)$ if the slit width is not less than a wavelength, and then, substituting the expression (11) for $g(\theta_o,\theta)$ we have finally

$$(21)\qquad \begin{aligned} C_a &= \left(-\frac{1}{\pi\sin(\theta_{bo}/2)} + \frac{e^{-2ikd}}{\pi\sin(\theta_{ao}/2)\sqrt{4\pi ikd}} \right) \left(1 - \frac{e^{-i4kd}}{4\pi ikd} \right)^{-1} \\ C_b &= \left(-\frac{1}{\pi\sin(\theta_{ao}/2)} + \frac{e^{-2ikd}}{\pi\sin(\theta_{ao}/2)\sqrt{4\pi ikd}} \right) \left(1 - \frac{e^{-i4kd}}{4\pi ikd} \right)^{-1} . \end{aligned}$$

The field representation which includes the interaction term should be a better approximation than the 'non-interaction' representation, and hence may be expected to hold for smaller slit widths than the latter. It also seems to have an advantage over the well-known solution in the form of an infinite series of Mathieu functions, in that it is an intuitively clear representation in terms of known elementary functions.

For convenience of expression the results derived in this section have been given in terms of a double set of coordinates. For computation, however, it is more convenient to express these results in a single coordinate system with origin midway between the half-planes (see Fig. 5).

Because the equations of transformation between the various systems of angular coordinates described above are different for $y > 0$ and $y < 0$, the case where the field point and the incident wave are on the same side of the plane $y = 0$ cannot be considered together with the case where the field point and incident wave are on opposite sides. Only the latter case will be considered here, and will yield the field transmitted by the slit.

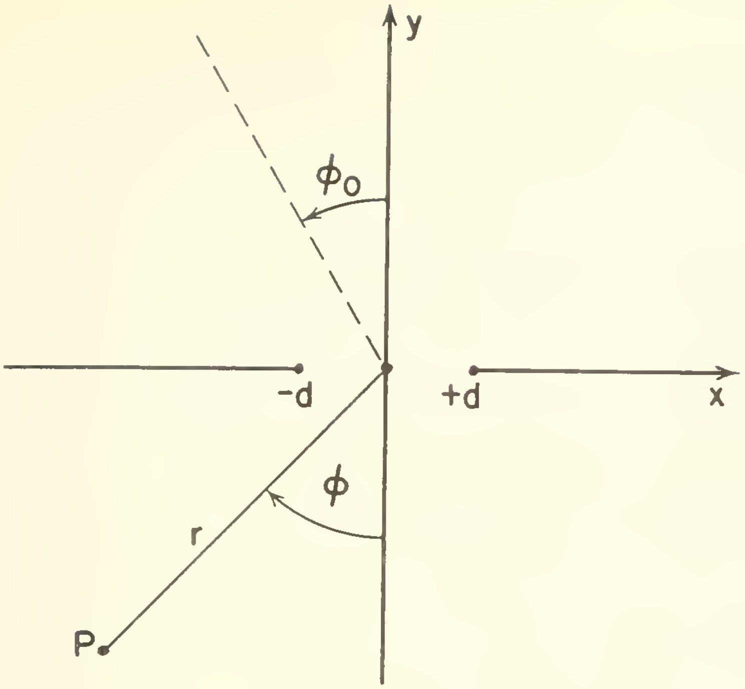

Figure 5. Single coordinate system. All angles are the acute angles measured from the nearest normal and are considered positive if measured counter-clockwise with respect to this normal, and negative if clockwise.

Let the incoming plane wave be incident from above the slit and the field point P be below. Further, let ϕ_o represent the acute angle between the normal to the incoming plane wave and the normal to the plane of the screen (measured from the positive y-axis), and let (r,ϕ) represent the field point P, where ϕ is the acute angle between the field point and the normal to the screen (measured from the negative y-axis). All angles are considered positive if measured counterclockwise with respect to the normal and negative if clockwise. The relations between these coordinates and those in each of the systems previously used is, in general, quite complicated, but when P is far from the edges as compared to the slit width, i.e., when

$$\frac{r}{2d} >> 1,$$

approximate relations between them can be simply stated. Remembering the convention that angles measured counter-clockwise are positive and angles measured clockwise are negative, we have

(22)

a) $r_a \simeq r + d \sin \phi$ b) $r_b \simeq r - d \sin \phi$

c) $\theta_a \simeq \frac{3}{2}\pi - \phi$ d) $\theta_b \simeq \frac{3}{2}\pi + \phi$

e) $\theta_{ao} = \frac{\pi}{2} - \phi_o$ f) $\theta_{bo} = \frac{\pi}{2} + \phi_o$.

The difference between (c), (d) and (e), (f) is due to the fact that ϕ is measured from the negative y-axis, while ϕ_o is measured from the positive y-axis.

Substituting the relations (22) into (11) yields

$$(23)\quad \begin{aligned} g(\theta_{ao},\theta_a) &\simeq -\frac{1}{2\pi}\left\{\frac{1}{\sin\frac{1}{2}(\phi-\phi_o)} + \frac{1}{\cos\frac{1}{2}(\phi+\phi_o)}\right\}, \\ g(\theta_{bo},\theta_b) &\simeq -\frac{1}{2\pi}\left\{\frac{-1}{\sin\frac{1}{2}(\phi-\phi_o)} + \frac{1}{\cos\frac{1}{2}(\phi+\phi_o)}\right\}. \end{aligned}$$

Similarly, substituting (22) into (6) yields

$$(24)\quad \begin{aligned} f(2d,\pi,\theta_a) &\simeq e^{+i2kd\sin\phi}\left\{-1+\sqrt{2i}\int_0^{2\sqrt{kd/\pi}\,[\cos(\phi/2)+\sin(\phi/2)]}\exp\left[-i\pi v^2/2\right]dv\right\}, \\ f(2d,\pi,\theta_b) &\simeq e^{-i2kd\sin\phi}\left\{-1+\sqrt{2i}\int_0^{2\sqrt{kd/\pi}\,[\cos(\phi/2)-\sin(\phi/2)]}\exp\left[-i\pi v^2/2\right]dv\right\}. \end{aligned}$$

For the values of θ where the approximation (8) is valid, i.e., away from the shadow lines $\theta_a = 0$ and $\theta_b = 0$, these expressions reduce to

$$(25)\quad \begin{aligned} f(2d,\pi,\theta_a) &\simeq -\frac{e^{-i2kd}}{\sqrt{2\pi ikd}}\,\frac{1}{\cos\frac{\phi}{2}+\sin\frac{\phi}{2}}, \\ f(2d,\pi,\theta_b) &\simeq -\frac{e^{-i2kd}}{\sqrt{2\pi ikd}}\,\frac{1}{\cos\frac{\phi}{2}-\sin\frac{\phi}{2}}. \end{aligned}$$

Finally, the coefficients C_a and C_b become

$$C_a = \frac{\sqrt{2}/\pi}{1 - \frac{e^{-i4kd}}{4\pi ikd}} \left\{ -\frac{1}{\cos(\phi_o/2) + \sin(\phi_o/2)} + \frac{1}{\cos(\phi_o/2) - \sin(\phi_o/2)} \frac{e^{-i2kd}}{\sqrt{4\pi ikd}} \right\},$$

(26)

$$C_b = \frac{\sqrt{2}/\pi}{1 - \frac{e^{-i4kd}}{4\pi ikd}} \left\{ -\frac{1}{\cos(\phi_o/2) - \sin(\phi_o/2)} + \frac{1}{\cos(\phi_o/2) + \sin(\phi_o/2)} \frac{e^{-i2kd}}{\sqrt{4\pi ikd}} \right\}.$$

Actually we have not verified that our field has the required asymptotic behavior for large kr [as specified in the conditions (ii) and (iii)], i.e., that in terms of the coordinates of Figure 5 we have, for $r \rightarrow \infty$,

$$E_z - E_z^{inc} - E^{reflected} \sim (\pi kr)^{-1/2} \exp(-ikr) F(\phi), \qquad y > 0 ,$$

and

$$E_z \sim (\pi kr)^{-1/2} \exp(-ikr) F(\phi), \qquad y < 0$$

uniformly in ϕ. To verify this behavior we recall the representation given in (13c) - (13e). In those equations $E_z(\text{non})$ is the leading term of Schwarzschild's solution. The asymptotic form of this term causes some difficulty at $\phi = \frac{\pi}{2}$, for a reason shortly to be indicated. However, Schwarzschild showed that it has the behavior we require [he did this in the course of obtaining an equation equivalent to our equation (27)]. The interested reader is therefore referred to ref. 8 for a proof. Now the cause of the difficulty mentioned above is that for plane-wave excitation the cylindrical wave representation [cf. Eq. (10)] for the diffracted fields breaks down near shadow lines, and the function $g(\theta_o, \theta)$ appearing in Eqs. (11), (13a) and (13b) tends to infinity [cf. the restriction as to direction given after Eqs. (13a) and (13b)]. On the other hand, no shadow-line phenomenon occurs in connection with the line-source response, so that the term $E_z(\text{int})$ in Eqs. (13a) and (13b) can be readily shown to have the proper asymptotic behavior, by simply employing the relations (22). In the next section the relevant calculation is carried out for the case of a normally incident wave.

It seems worthwhile to remark that in treating E_z(non), if we pass to the limit $r \to \infty$ with $\phi \neq \phi_o$, thus avoiding the shadow-line direction, and then let $\phi \to \phi_o$, the latter limit is readily found to exist. This is a consequence of the fact that under these circumstances θ_a and θ_b approach the shadow line from lit and unlit sides respectively, so that the divergent parts are cancelled. The situation is illustrated in Eq. (27) of the next section.

IV. Numerical results on the normally incident plane wave

To apply the results of the previous section to the special case of a plane wave normally incident, it is necessary only to set $\phi_o = 0$. Making this substitutiong we find from (23), (22a), (22b) and (15a) that the non-interaction term of the far transmitted field for this case is, for $r \to \infty$, with $\phi \neq \frac{\pi}{2}$,

$$E_z^{(d)}(\text{non}) \cong \sqrt{\frac{\pi}{2i}}\,\frac{e^{-ikr}}{\sqrt{kr}}\left\{-\frac{e^{-ikd\sin\phi}}{2\pi}\left(\frac{1}{\sin\frac{\phi}{2}}+\frac{1}{\cos\frac{\phi}{2}}\right)\right.$$

$$(27)\qquad \left.-\frac{e^{+ikd\sin\phi}}{2\pi}\left(-\frac{1}{\sin\frac{\phi}{2}}+\frac{1}{\cos\frac{\phi}{2}}\right)\right\},$$

which can be written in the somewhat simpler form

$$(28)\qquad E_z^{(d)}(\text{non}) \cong \frac{e^{-ikr}}{\sqrt{\pi kr}}\;\frac{\sin(kd\sin\phi-\frac{\phi}{2})+i\sin(kd\sin\phi+\frac{\phi}{2})}{\sin\phi}\;.$$

Further, if we set $\phi_o = 0$ in (26) the coefficients simplify to

$$(29)\qquad C_a = C_b = \frac{-\sqrt{2}/\pi}{1+\dfrac{e^{-i2kd}}{\sqrt{4\pi ikd}}}\;.$$

Now we substitute (29) and (24) into (15b) and obtain the interaction contribution to the far transmitted field:

$$E_z^{(d)}(\text{int}) = -\frac{e^{-ikr}}{\sqrt{\pi kr}} \cdot \frac{e^{-i\pi/4}}{1+\frac{e^{-i2kd}}{\sqrt{4\pi ikd}}} \left\{ e^{+ikd\sin\phi}\left[-1+\sqrt{2i}\int_0^{2\sqrt{kd/\pi}\,[\cos(\phi/2)+\sin(\phi/2)]} \exp(-i\pi\, v^2/2)dv\right]\right.$$

$$(30) \qquad \left. + \; e^{-ikd\sin\phi}\left[-1+\sqrt{2i}\int_0^{2\sqrt{kd/\pi}\,[\cos(\phi/2)-\sin(\phi/2)]} \exp(-i\pi\, v^2/2)dv\right]\right\} .$$

When (25) is applicable, i.e., away from the lines $\phi = \pm \frac{\pi}{2}$, this equation reduces to

$$(31) \qquad E_z^{(d)}(\text{int}) = \frac{e^{-ikr}}{\sqrt{\pi kr}} \left\{ \frac{2e^{-i2kd}}{\sqrt{4\pi ikd} + e^{-i2kd}} \; \frac{\cos(kd\sin\phi - \frac{\phi}{2}) - i\cos(kd\sin\phi + \frac{\phi}{2}}{\cos\phi} \right\} .$$

The above equations enable us to calculate the function $F(\phi)$ defined by the equation

$$(31a) \qquad E_z = E_z^d \sim (\pi kr)^{-1/2} \exp(-ikr)F(\phi), \qquad (r \to \infty,\ y < 0).$$

Let us now calculate $F(\phi)$ in the limit $\phi \to 0$. Using (28) and (31) we find

$$(31b) \qquad F(0) \cong \frac{1}{2}\left\{2kd - 1 + i(2kd+1)\right\} + \frac{2(1-i)\exp(-2ikd)}{\sqrt{4\pi ikd} + \exp(-2ikd)} .$$

Use of equation (30) yields a somewhat more accurate result. For wave functions vanishing on the screen it is well known that there is a simple relation between $F(0)$ and the transmission coefficient T. One can show that

$$(31c) \qquad T = \frac{\mathrm{Re}(1-i)F(0)}{2kd} = \frac{2\int_0^{\pi/2} |F|^2 d\phi}{2\pi kd} .$$

Since our function u is only approximately correct and only approximately satisfies the boundary conditions, we have calculated T by both methods indicated in equation (31c).

To determine the accuracy of the approximation procedure developed in this paper we shall use equations (14), (28), (29) and (31) to derive numerical values for the field strengths and then compare the results with the values computed from the exact solutions in terms of Mathieu function. From the exact solution, the far diffracted field transmitted by a slit when excited by a plane wave normally incident can be written as a series in Mathieu functions[1,2,3]:

$$\lim_{r\to\infty} E_z = \frac{e^{-ikr}}{\sqrt{\pi kr}} \left\{ \sqrt{\pi i} \sum_r (-1)^r A_{2r+1} So_{2r+1}(k^2d^2, \emptyset + \frac{\pi}{2}) \right\}, \tag{32}$$

where

$$A_{2r+1} = \frac{kd(B_{2r+1})^2 g_{o,2r+1} Do_1^{(2r+1)}}{1 - if_{o,2r+1}} . \tag{33}$$

The notation used here is that employed in the Bureau of Standards Tables[10] and the definitions agree with those of Stratton[12]. Equations (32) and (33) have been derived in a straightforward manner following the procedures of Skavlem[3], except for a difference in notation. The far-field form (32) is obtained by letting the elliptic coordinate corresponding to r go to infinity and factoring out the common asymptotic form of the functions of that coordinate.

The usefulness of the exact solution (32), (33) is limited to slit widths where kd < 10, since from very large slits too many terms in the expansion (32) must be considered. On the other hand, the approximation (28), (30) is valid only for wide slits (kd large), where each edge is in the far field of the opposite edge. Fortunately, there exists a region of overlap ($3 \le kd \le 10$) in which the approximate solution can be compared with the exact one. In this range of slit sizes, twelve values were chosen for which the coefficients of the Mathieu functions* are tabulated in the Bureau of Standards tables, so that

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* The Mathieu functions themselves are not tabulated as functions of the arguments, but only the coefficients $Do_{2k+1}^{(2r+1)}(k^2d^2)$ from which the Mathieu functions are determined by the formula

$$So_{2r+1}(k^2d^2, x) = \sum_{k=o}^{\infty} Do_{2k+1}^{(2r+1)}(k^2d^2)\sin(2k+1)x.$$

interpolation was avoided. The various quantities appearing in the formulas (32) and (33) were taken correct to five significant figures and the intermediate results rounded out to four; the far-field amplitude $F(\phi)$ is given to three decimal places. A spot check reveals, in some instances, errors of one or two units in the last decimal place; this is due to the large number of operations involved.

In order to check the accuracy of our approximate solution, the far-field amplitude $F(\phi)$ was calculated in accordance with our approximation. For an understanding of the tables subsequently to be presented one must bear in mind that these approximate computations are intended only for purposes of comparison with the exact solution, and that discrepancies are in most cases apparent already in the second decimal place. It should also be noted that our formula (30) involves the Fresnel integral. For large enough kd this integral can be approximated in terms of elementary functions, provided the angle ϕ is not too close to 90°. In order to control this behavior, we started our computations with the angle $\phi = 0$, and, using the elementary approximation to the Fresnel integral, we proceeded by 10° increments, comparing each result with the Mathieu-function result. When our level of accuracy decreased sharply we reverted to the use of the exact Fresnel integral, working, however, only to two decimal places. This was found adequate to reduce the discrepancy to the same level as that obtained for smaller angular values of ϕ. For $\phi = 90^\circ$ the result to two decimal places was $F(\phi) = 0$ (except for the smallest kd we used); this is in agreement with the boundary conditions.

Table I lists the values of the far field at 10° intervals from $\phi = 0^\circ$ to $\phi = 90^\circ$ for: a) the exact solution (32), (33); b) the non-interaction approximation (28); and c) the approximation corrected for the interaction term (30). All field strengths are divided by the radial factor $e^{-ikr}/\sqrt{\pi kr}$. It is seen from Table I that the interaction term can by no means be neglected in this range of slit sizes.

In addition to the calculation of the far fields we also computed the transmission coefficient

$$(34) \qquad T = \int_0^{\pi/2} |F|^2 d\phi \Big/ kd.$$

For the exact $F(\phi)$ this quantity is also given by the formula

$$(34a) \qquad T = \mathrm{Re}(1-i)F(0) \Big/ 2kd,$$

but the above formulas need not necessarily agree if $F(\phi)$ is incorrect. In our computations we calculated approximate values of T by the trapezoidal rule from the approximate $F(\phi)$ of Table I, using equation (34). To control the intrinsic error due to numerical integration we also performed the same calculation using the exact values of $F(\phi)$ from Table I, and compared these with the values obtained from Table I via equation (34). The discrepancy was not more than .001. (Use of Simpson's rule instead of the trapezoidal rule was found to produce greater errors in some cases.) In addition to these results, it seemed of interest to compile the values which are obtainable using our approximate solution in conjunction with formula (34a). In this connection F(0) may be obtained using equation (31). This leads to equation (31b), and so to the formula

$$(34b) \qquad T = 1 + \mathrm{Re}\left[2(1-i)^2\exp(-2ikd)\right]\left[1 + \frac{\exp(-2ikd)}{\sqrt{4\pi ikd}}\right]^{-1}\left[2kd\sqrt{4\pi ikd}\right]^{-1}.$$

The approximate value of F(0) involved here is obtained by using an asymptotic evaluation of the Fresnel integral occurring in (30), and it should therefore be compared with the values obtained by numerical integration from Table I. A more accurate reflection of our approximate solution is obtained by using equation (30) itself in conjunction with (34a). This gives the approximate formula

$$(34c) \qquad T = 1 + \frac{1}{2kd} \frac{\mathrm{Re}\left[4\exp(-i\pi/4)\right]\left[\int_{2\sqrt{kd/\pi}}^{\infty} \exp(-\frac{i\pi v}{2})dv\right]}{1 + (4\pi kd)^{-1/2}\exp\left[-i(2kd+\frac{\pi}{4})\right]}$$

which turns out to be in error by (-0.005) for kd = 3, i.e., when the slit width 2d is $3/\pi$ times the wavelength λ. In contrast, the simpler formula (34b) is in error by (-0.012) for this value of kd.

Of special interest is the asymptotic formula obtainable when we retain only the leading term of (34b). We find

$$(34d) \qquad T(\text{asymptotic}) = 1 - \frac{1}{\sqrt{\pi}\,(kd)^{3/2}} \cos(2kd-\pi/4).$$

This simple result turns out to be in error by (-0.024) at kd = 3. For kd = 8 however, the error is (-0.001) which should be compared with the error of (+0.001) resulting from all the previous formulas. All the quantities we have described here are given in Table II, where we have also included the values given by Skavlem for $3 \leq kd \leq 8$.

It should be noted that the accuracy of the approximate transmission coefficient exceeds the accuracy of the far-field pattern. This is due to the fact that in computing the transmission coefficient from the forward amplitude the error in the latter is divided by 2kd. A better conception of the accuracy is formed if one considers the percentage error of the deviation from unity. For kd = 3 the percentage error of the deviation from unity of (34c), (34b) and (34d) are, respectively, 17%, 41% and 82%. For kd = 8 all the formulas differ by about .001 from the true value of 1.023. Thus the percentage error is of the order 1/23 or 4%. Actually, if we refrain from rounding off, we obtain, instead, percentage errors of 1%, 2% and 4% for kd = 8 (cf. Table II).

It seems useful to present at this point a brief analysis of the significance of the successive simplifications which lead from equation (34c), through (34b), to equation (34d). It will be recalled that we take account of interaction in our theory by regarding each half-plane as responding to a secondary excitation. This secondary excitation is taken to be a line source, and the strength of the line source is then itself determined by taking account of the interaction. The secondary response of each half-plane is thus a product of two factors; it is the response to a unit line source multiplied by an assigned source strength. One may now approximate each of these factors. First of all we can ignore interaction in determining the strength of the line source. The far-field form of the response of each half-plane to the incident wave itself has the form of a line-source field. This line-source field is then regarded as the secondary excitation for the other half-plane. An additional simplification is possible in calculating the far-field form of the secondary response to a line source. It will be shown that this is effectively the same as the response to a plane wave. In fact, the far-field angular amplitude in a certain direction θ, say, can be expressed, by reciprocity, as the response to a plane wave coming from the direction θ. The latter response is to be evaluated at the source point. This statement is strictly correct so far. But when the source point is far from the edge of the excited half-plane, we now have available the simplification involved in replacing the plane-wave response at the source point by its far-field form (provided the direction θ is not too close to the responding half-plane, in which case the horizontal direction would be near a shadow line). We will then be calculating the far-field response in the horizontal direction due to an incident plane wave. This can be expressed, again by reciprocity, in terms of the far-field response of the half-plane, in the direction θ, due to excitation by a plane wave traveling horizontally, i.e., coming from the direction of the opposite edge. Thus in the far field the response of a half-plane to a distant line source is effectively the response to a plane wave of suitable strength, despite the fact that the excitations differ markedly along the half-plane except in the neighborhood of the edge, and the responses differ markedly except in the far-field. This phenomenon is presumably connected with the fact that for large kr and kd the wave number k is large in effect, so that geometric-optical considerations are valid. Let us now return to consideration of the interaction correction to the transmission coefficient. Since this is calculated from the forward far-field amplitude it seems that both kinds of simplification we have described are available. The transition from Equations (34c) through (34b) to (34d) can now be explained conceptually. In (34c) the second excitation is a line source. Its strength is derived from interaction consideration, and then slightly simplified.

In (34b) this line source has been effectively replaced by the equivalent plane wave. The influence of interaction upon the strength of the secondary excitation is still retained. The same is true in (34d), but in addition the effect of interaction on the strength of the (simplified) secondary excitation has been obliterated; the influence of interaction is observed only in the secondary response. Table II shows the wavelength dependence of decrease in accuracy entailed by the simplifications in question. Hence from this table one may form a conception of the relative usefulness of the various idealizations we have been discussing.

V. Excitation by a line source

When the slit is excited by a line source instead of a plane wave, an approximation to the field can be obtained by the same method as that used in Section III, except, of course, that now each half-plane is excited by two line sources instead of a plane wave and a line source. We append here a brief discussion of this case.

In this section we use the new notation $\left[E_z^{(r)}(r_o,\theta_o,r,\theta)\right]_i$ for the exact response field produced when an isolated half-plane is excited by a line source (r_o,θ_o). This field is found by subtracting from the total field (2) the corresponding incident field (see Fig.2). As noted earlier*, the far response field can be expressed asymptotically and uniformly as an outgoing wave whose amplitude is a function of (r_o,θ_o,θ). The (non-uniform) asymptotic expression for the diffracted field, on the other hand, is

$$\left[E_z^{(d)}(r_o,\theta_o,r,\theta)\right]_i \simeq \sqrt{\frac{\pi}{2i}}\ \frac{e^{-ikr}}{\sqrt{kr}}\ f(r_o,\theta_o,\theta) \quad \text{for} \quad kr \gg 1. \tag{35}$$

Previously we have used the function $f(r_o,\theta_o,\theta)$ for the special case $\theta_o = \pi$. Away from the two shadow lines $\theta = \pi \pm \theta_o$ this function is given, for $kr_o \gg 1$, by the asymptotic form

$$f(r_o,\theta_o,\theta) \simeq -\frac{1}{2\sqrt{2\pi i}}\ \frac{e^{-ikr_o}}{\sqrt{kr_o}} \left\{ \frac{1}{\cos\frac{1}{2}(\theta-\theta_o)} - \frac{1}{\cos\frac{1}{2}(\theta+\theta_o)} \right\}, \tag{35a}$$

as can be shown from Equation (2). This approximation breaks down in an angular sector surrounding each shadow line - a narrow sector if kr_o is large and a wide sector if kr_o is small.

Since half-plane (a) is excited by line sources at (r_{ao},θ_{ao}) and at $(2d,\pi)$ in the coordinate system associated with half-plane (a), while half-plane (b) is excited by line sources at (r_{bo},θ_{bo}) and at $(2d,\pi)$ in the coordinate system associated with half-plane (b), the total field E_z is given by the sum

$$\begin{aligned} E_z \simeq & \left\{ \left[E_z^{(d)}(r_{ao},\theta_{ao},r_a,\theta_a)\right]_i + \left[E_z^{(d)}(r_{bo},\theta_{bo},r_b,\theta_b)\right]_i \right\} + E_z^{inc} \\ & + \left\{ C_a'\left[E_z^{(d)}(2d,\pi,r_a,\theta_a)\right]_i + C_b'\left[E_z^{(d)}(2d,\pi,r_b,\theta_b)\right]_i \right\} \\ = & \ E_z^{(d)}(\text{non}) + E_z^{(d)}(\text{int}). \end{aligned} \tag{36}$$

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* Cf. Eq. (4a).

A similar equation could have been written in connection with Eq. (13) of Section III. Eq. (36) holds approximately for any point, far or near, as long as the slit is wide enough. We can now evaluate the terms of (36) for large kr, in accordance with (35) and (35a), and get an equation of the same type as Eq. (13). The coefficients C_a' and C_b' which give the strength of the interaction term are found, by the method of Section III, to be

$$C_a' \simeq \frac{f(r_{bo},\theta_{bo},\pi) + f(r_{ao},\theta_{ao},\pi)f(2d,\pi,\pi)}{1 - \left[f(2d,\pi,\pi)\right]^2}$$

(37)

$$C_b' \simeq \frac{f(r_{ao},\theta_{ao},\pi) + f(r_{bo},\theta_{bo},\pi)f(2d,\pi,\pi)}{1 - \left[f(2d,\pi,\pi)\right]^2}$$

for large slit widths ($kd >> 1$).

The introduction of a single coordinate system such as employed in Section III can be used to simplify the results. A further simplification ensues if we have a source remote from the slit, i.e., if $kr_o >> 1$. In addition it seems useful to remark that the calculation of far-fields can be considerably simplified by the use of the reciprocity theorem. Even in the presence of screens the latter theorem, in a limiting form, can be used to express the far-field of a line source, in terms of the near field of a plane wave. In consequence of our approximations the latter field for the slit problem can be expressed in terms of Fresnel integrals.

Conclusion

In the foregoing work we have derived a simple and convenient expression for the field diffracted by an infinite slit in a conducting plane, i.e., when the wavelength is greater than or equal to the slit width. The principal result is that this field can be accurately calculated everywhere by considering each half-plane composing the screen to be excited by the incident field, and, in addition, by a 'virtual' line source located at the edge of the opposite half-plane [Eqs. (13) - (13e)]. The strength of this line source can be simply expressed in terms of trigonometrical functions and Fresnel integrals which depend upon the ratio of slit width to wavelength, [Eqs. (20), (6) and (11)]. These expressions may be further simplified for very large kd, but should be retained intact for use with slits of the order of one or two wavelengths [cf. conclusion of Section IV, and Table II].

Up to the present, calculations for moderate to large slit widths have required the use of the Mathieu function theory. The convergence of such calculations becomes very slow as the slit width increases to three wavelengths. In addition only the case of a normally incident plane wave has been considered numerically in the literature. On the other hand, the present procedure permits a) a rapid calculation, for slit widths greater than one wavelength; b) the accuracy and computational convenience increases with slit width; c) the procedure is capable of yielding both near and far fields in terms of the known solutions for a half-plane; d) the method is equally well applicable to more general excitations (e.g., oblique plane wave or line source).

Table I

Legend: The first column, headed $F_{ex}(\phi)$, gives the exact angular dependence of the far field. [Eq. (32)]. The second column, headed $F_{non}(\phi)$, gives the angular dependence calculated from the non-interaction approximation. The third column gives the angular dependence calculated from the approximation developed in the present paper, in which interaction between half-planes is taken into account [Eqs. (30),(31)].

$(kd)^2 = 9$

ϕ	$F_{ex}(\phi)$	$F_{non}(\phi)$	$F_{non}(\phi) + F_{int}(\phi)$
0	2.678 + 3.154i	2.500 + 3.500i	2.639 + 3.122i
10	2.589 + 2.997i	2.420 + 3.291i	2.558 + 2.966i
20	2.343 + 2.574i	2.20 + 2.726i	2.33 + 2.543i
30	1.989 + 2.001i	1.890 + 1.964i	2.014 + 1.976i
40	1.587 + 1.413i	1.556 + 1.184i	1.61 + 1.42 i
50	1.19 + .908i	1.251 + .518i	1.22 + .93 i
60	.830 + .531i	1.012 + .023i	.85 + .54 i
70	.518 + .278i	.855 - .303i	.53 + .30 i
80	.248 + .16 i	.786 - .496i	.26 + .13 i
90	.000 + .000i	.800 - .600i	.00 + .01 i

$(kd)^2 = 10$

ϕ	$F_{ex}(\phi)$	$F_{non}(\phi)$	$F_{non}(\phi) + F_{int}(\phi)$
0	2.732 + 3.278i	2.662 + 3.662i	2.688 + 3.260i
10	2.643 + 3.096i	2.565 + 3.422i	2.606 + 3.076i
20	2.395 + 2.612i	2.30 + 2.780i	2.38 + 2.586i
30	2.035 + 1.967i	1.937 + 1.926i	2.057 + 1.938i
40	1.627 + 1.323i	1.546 + 1.071i	1.65 + 1.34 i
50	1.22 + .793i	1.195 + .365i	1.25 + .82 i
60	.852 + .421i	.924 - .139i	.86 + .44 i
70	.532 + .195i	.750 - .454i	.55 + .22 i
80	.255 + .072i	.674 - .631i	.26 + .08 i
90	.000 + .000i	.692 - .721i	.00 + .00 i

$(kd)^2 = 12$

ϕ	$F_{ex}(\phi)$	$F_{non}(\phi)$	$F_{non}(\phi) + F_{int}(\phi)$
0	2.812 + 3.591i	2.964 + 3.964i	2.765 + 3.599i
10	2.725 + 3.344i	2.833 + 3.660i	2.685 + 3.346i
20	2.480 + 2.694i	2.478 + 2.859i	2.462 + 2.681i
30	2.122 + 1.865i	1.990 + 1.824i	2.133 + 1.831i
40	1.706 + 1.083i	1.483 + .834i	1.72 + 1.07 i
50	1.285 + .499i	1.042 + .069i	1.32 + .50 i
60	.898 + .148i	.713 - .431i	.92 + .14 i
70	.561 - .007i	.508 - .705i	.59 - .01 i
80	.268 - .034i	.422 - .836i	.28 - .02 i
90	.000 + .000i	.446 - .895i	.00 - .01 i

$(kd)^2 = 16$

ϕ	$F_{ex}(\phi)$	$F_{non}(\phi)$	$F_{non}(\phi) + F_{int}(\phi)$
0	3.064 + 4.474i	3.500 + 4.500i	3.053 + 4.523i
10	2.951 + 4.013i	3.286 + 4.058i	2.937 + 4.050i
20	2.639 + 2.841i	2.719 + 2.923i	2.621 + 2.847i
30	2.193 + 1.446i	1.972 + 1.541i	2.173 + 1.409i
40	1.696 + .292i	1.237 + .342i	1.70 + .26 i
50	1.22 - .396i	.641 - .458i	1.24 - .43 i
60	.813 - .633i	.231 - .865i	.84 - .67 i
70	.486 - .557i	-.007 - 1.002i	.50 - .56 i
80	.225 - .311i	-.101 - 1.013i	.24 - .32 i
90	.000 + .000 i	-.073 - .997i	.00 + .00 i

$(kd)^2 = 20$

ϕ	$F_{ex}(\phi)$	$F_{non}(\phi)$	$F_{non}(\phi) + F_{int}(\phi)$
0	3.755 + 5.337i	3.972 + 4.972i	3.790 + 5.364i
10	3.484 + 4.625i	3.663 + 4.378i	3.507 + 4.649i
20	2.785 + 2.866i	2.856 + 2.898i	2.776 + 2.881i
30	1.915 + .894i	1.840 + 1.200i	1.873 + .888i
40	1.13 - .570i	.898 - .128i	1.10 - .58 i
50	.564 - 1.261i	.197 - .861i	.56 - 1.26 i
60	.299 - 1.309i	-.238 - 1.098i	.22 - 1.33 i
70	.068 - .988i	-.462 - 1.059i	.08 - .99 i
80	.012 - .514i	-.543 - .939i	.02 - .52 i
90	.000 + .000i	-.519 - .855i	.00 + .00 i

$(kd)^2 = 26$

ϕ	$F_{ex}(\phi)$	$F_{non}(\phi)$	$F_{non}(\phi) + F_{int}(\phi)$
0	4.852 + 5.836i	4.599 + 5.599i	4.874 + 5.818i
10	4.268 + 4.927i	4.123 + 4.759i	4.284 + 4.918i
20	2.831 + 2.745i	2.924 + 2.749i	2.834 + 2.759i
30	1.24 + .461i	1.507 + .647i	1.216 + .490i
40	.054 - 1.03 i	.328 - .726i	.04 - 1.03 i
50	-.533 - 1.552i	-.421 - 1.217i	-.53 - 1.55 i
60	-.647 - 1.398i	-.784 - 1.125i	-.64 - 1.40 i
70	-.505 - .958i	-.917 - .820i	-.51 - .96 i
80	-.265 - .469i	-.939 - .543i	-.27 - .45 i
90	.000 + .000i	-.922 - .389i	.00 + .00 i

$(kd)^2 = 30$

ϕ	$F_{ex}(\phi)$	$F_{non}(\phi)$	$F_{non}(\phi) + F_{int}(\phi)$
0	5.288 + 5.987i	4.977 + 5.977i	5.291 + 5.964i
10	4.558 + 4.989i	4.378 + 4.962i	4.563 + 4.977i
20	2.795 + 2.646i	2.900 + 2.597i	2.804 + 2.656i
30	.911 + .299i	1.233 + .280i	.912 + .330i
40	-.385 - 1.11 i	- .047 - 1.036i	-.38 - 1.10 i
50	-.926 - 1.483i	- .756 - 1.301i	-.94 - 1.49 i
60	-.926 - 1.237i	-1.017 - .982i	-.94 - 1.24 i
70	-.667 - .794i	-1.048 - .534i	-.66 - .80 i
80	-.335 - .369i	-1.015 - .193i	-.34 - .38 i
90	.000 + .000i	-1.000 - .021i	.00 + .00 i

$(kd)^2 = 36$

ϕ	$F_{ex}(\phi)$	$F_{non}(\phi)$	$F_{non}(\phi) + F_{int}(\phi)$
0	5.675 + 6.269i	5.500 + 6.500i	5.663 + 6.257i
10	4.806 + 5.103i	4.70 + 5.204i	4.80 + 5.092i
20	2.734 + 2.458i	2.787 + 2.317i	2.748 + 2.462i
30	.595 + .017i	.785 + .240i	.589 + .036i
40	- .781 - 1.21 i	- .557 - 1.360i	- .77 - 1.21 i
50	-1.260 - 1.334i	-1.111 - 1.239i	-1.26 - 1.33 i
60	-1.14 - .945i	-1.154 - .617i	-1.14 - .95 i
70	- .781 - .514i	-1.012 - .036i	- .79 - .52 i
80	- .382 - .209i	- .892 + .323i	- .38 - .22 i
90	.000 + .000i	- .876 + .481i	.00 + .00 i

$(kd)^2 = 42$

ϕ	$F_{ex}(\phi)$	$F_{non}(\phi)$	$F_{non}(\phi) + F_{int}(\phi)$
0	5.913 + 6.692i	5.981 + 6.981i	5.893 + 6.695i
10	4.954 + 5.263i	4.962 + 5.393i	4.947 + 5.261i
20	2.690 + 2.170i	2.604 + 1.992i	2.702 + 2.160i
30	.386 - .385i	.324 - .707i	.413 - .387i
40	-1.040 - 1.331i	- .973 - 1.526i	-1.04 - 1.33 i
50	-1.476 - 1.10 i	-1.283 - 1.009i	-1.47 - 1.09 i
60	-1.278 - .539i	-1.074 - .168i	-1.28 - .53 i
70	- .850 - .153i	- .767 + .432i	- .87 - .16 i
80	- .409 - .008i	- .573 + .726i	- .42 - .01 i
90	.000 + .000i	- .554 + .832i	.00 + .00 i

$(kd)^2 = 50$

ϕ	$F_{ex}(\phi)$	$F_{non}(\phi)$	$F_{non}(\phi) + F_{int}(\phi)$
0	6.251 + 7.524i	6.571 + 7.571i	6.248 + 7.547i
10	5.130 + 5.528i	5.234 + 5.570i	5.126 + 5.535i
20	2.542 + 1.523i	2.286 + 1.525i	2.543 + 1.505i
30	.052 - 1.134i	- .264 - 1.221i	.065 - 1.159i
40	-1.323 - 1.386i	-1.353 - 1.530i	-1.32 - 1.39 i
50	-1.581 - .456i	-1.259 - .546i	-1.57 - .45 i
60	-1.24 + .326i	- .730 + .411i	-1.26 + .34 i
70	- .761 + .541i	- .262 + .880i	- .76 + .56 i
80	- .345 + .352i	- .018 + .997i	- .36 + .35i
90	.000 + .000i	.003 + 1.000i	.00 + .00 i

$(kd)^2 = 56$

ϕ	$F_{ex}(\phi)$	$F_{non}(\phi)$	$F_{non}(\phi) + F_{int}(\phi)$
0	6.751 + 8.207i	6.983 + 7.983i	6.765 + 8.224i
10	5.311 + 5.701i	5.393 + 5.660i	5.313 + 5.707i
20	2.188 + .936i	2.007 + 1.159i	2.174 + .926i
30	- .415 - 1.680i	- .664 - 1.519i	- .422 - 1.704i
40	-1.449 - 1.267i	-1.507 - 1.403i	-1.45 - 1.28 i
50	-1.312 + .185i	-1.089 - .150i	-1.33 + .18 i
60	- .806 + 1.031i	- .371 + .762i	- .80 + 1.04 i
70	- .410 + 1.01 i	.146 + 1.041i	- .38 + 1.03 i
80	- .142 + .586i	.384 + .993i	- .15 + .59 i
90	.000 + .000i	.403 + .916i	.00 + .00 i

$(kd)^2 = 64$

ϕ	$F_{ex}(\phi)$	$F_{non}(\phi)$	$F_{non}(\phi) + F_{int}(\phi)$
0	7.596 + 8.779i	7.500 + 8.500i	7.610 + 8.773i
10	5.545 + 5.790i	5.552 + 5.732i	5.548 + 5.791i
20	1.491 + .406i	1.601 + .668i	1.475 + .413i
30	-1.13 - 2.020i	-1.123 - 1.801i	-1.142 - 2.025i
40	-1.395 - .991i	-1.551 - 1.107i	-1.39 - 1.01 i
50	- .592 + .742i	- .728 + .362i	- .60 + .74 i
60	.065 + 1.456i	.139 + 1.063i	.07 + 1.45 i
70	.271 + 1.23 i	.622 + 1.024i	.27 + 1.22 i
80	.188 + .639i	.793 + .762i	.18 + .64 i
90	.000 + .000i	.803 + .597i	.00 + .00 i

Table II

Transmission Coefficients

Legend: T_1 = Exact value from formula (34a) and Table I.

T_2 = Exact value from Skavlem[3]

T_3 = Value from numerical integration of exact values of the far field. (cf. first column of Table I)

T_4 = Value from numerical integration of approximate values of the far field (cf. last column of Table I)

T_5 = Value from formula (34c)

T_6 = Value from formula (34b)

T_7 = Value from formula (34d)

ka	T_1	T_2	T_3	T_4	T_5	T_6	T_7
3	0.972	0.97202	0.9714	0.9629	0.968	0.960	0.948
$\sqrt{10}$	0.950	-	0.9495	0.9425	0.948	0.940	0.932
3.4	-	0.92824	-	-	-	-	-
$\sqrt{12}$	0.924	-	0.9235	0.9180	0.920	0.919	0.913
4	0.942	-	0.9417	0.9466	0.944	0.947	0.951
$\sqrt{20}$	1.017	-	1.0157	1.0223	1.020	1.023	1.018
5	-	1.04992	-	-	-	-	-
$\sqrt{26}$	1.048	-	1.0478	1.0488	1.047	1.048	1.049
$\sqrt{30}$	1.029	-	1.0288	1.0288	1.028	1.027	1.030
6	0.995	0.99559	0.9948	0.9928	0.995	0.993	0.991
$\sqrt{42}$	0.972	-	0.9722	0.9711	0.972	0.971	0.969
7	-	0.97174	-	-	-	-	-
$\sqrt{50}$	0.974	-	0.9735	0.9755	0.975	0.975	0.979
$\sqrt{56}$	0.9994	-	0.9987	1.0020	1.0005	1.0015	1.0088
8	1.0234	1.02332	1.0233	1.0240	1.0236	1.0239	1.0220

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